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THE MAXIMUM PRINCIPLE FOR POSITIONAL CONTROLS AND THE PROBLEM OF OPTIMAL SYSTEM SYNTHESIS†

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An analogue of Pontryagin's maximum principle is derived in the class of positional controls (functions of the phase state and the time [1]). Corresponding adjoint first-order linear partial differential equations are presented. The use of the necessary conditions of optimality to synthesize optimal systems is described.

The well-known theoretical approaches to optimal system synthesis include the maximum principle [2], dynamic programming [3], the sufficient optimum conditions [4] and the theory of fields of extremals [5]. Each approach uses premises and constructions that determine and sometimes also limit its area of application. These are, for example, the assumption in [3] that the Bellman function is smooth, the uncertainty in the choice of auxiliary functions in [4] and the need in [2, 5] to solve a family of optimal open-loop control problems with arbitrary initial values of the trajectories.

In this paper we will present an analogue of Pontryagin's maximum principle [2] directly in the class of positional controls. As the object of the investigation we have chosen the relatively simple terminal control problem with free right end of the trajectory. To obtain the necessary conditions of optimality we use the technique of spiked variation of the control and small variation of its surfaces of discontinuity. At the stage when the principal terms of the increment to the object functional are being represented one derives the adjoint linear system of partial differential equations with appropriate boundary conditions – the so-called adjoint boundary-value problem. On the assumption that the adjoint boundary-value problem possesses a continuous piecewise-smooth solution, it is shown that an optimal positional control maximizes the Hamiltonian.

By analogy with [2], we shall call our result the positional maximum principle. We shall also show how this result is related to Pontryagin's maximum principle and to the fundamental equation of dynamic programming, and extend it to the case when the control has several surfaces of discontinuity. The possibilities of the positional maximum principle are demonstrated for linear and linear-quadratic optimal control problems. A procedure is proposed for synthesizing a piecewise-constant control, and an illustrative example is considered.

Some clarification is in order concerning our basic assumptions and the procedure employed. The maximum principle will be proved, for simplicity, for a positional control with one surface of discontinuity. It is assumed that the trajectories generated by the control either cut the surface of discontinuity without unilateral tangency or remain upon it. This pattern is characteristic for many of the synthesis problems that have been solved [2, 6]. The results obtained for a single surface of discontinuity can be easily extended to the case of several such surfaces, provided the latter do not intersect in the part of space under consideration. The appropriate generalizations will be stated without proof.

1. STATEMENT OF THE PROBLEM

Consider the optimal control problem

$$J=\Phi(x(t_1))\rightarrow \min,\quad \dot{x}=f(x,u,t)$$
$$u(x,t)\in U,\quad (x,t)\in R^n\times T \tag{1.1}$$

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where Φ: $R^n \to R$ is differentiable, f: $R^n \times U \times T \to R^n$ is continuous together with the matrices of derivatives f_x and f_u of the function, U is a non-empty bounded subset R^r and $T = [0, t_1]$ is a fixed closed interval on the real axis.

A piecewise-continuous function u: $R^n \times T \to U$ with piecewise-continuous derivative u_x is called a control. A solution of the vector differential equation (1.1) for a given control and initial values are defined as in Filippov [7]. If the initial values lie in the domain of smoothness of the control, the solution is identical in a small neighbourhood of the initial values with the classical solution, and it exists and is unique [8] by virtue of well-known results in the qualitative theory of differential equations. For other initial data the solution need not be unique. In that case we shall associate with the control $u(x, t)$ and the initial values $x_0 \in R^n$, $t_0 \in T$ one of the solutions $x(t, x_0, t_0)$ of the differential equation (1.1) for which the function $\Phi(x(t_1, x_0, t_0))$ has its least value. All the solutions are assumed to be extendible to the interval T, whatever the control.

Let $u(x, t)$ be an arbitrary control, which will remain fixed throughout. With it we associate the set $D \subset R^n \times T$ filled out by the corresponding integral curves of Eq. (1.1). This means that, together with any point (x_0, t_0), D will contain the entire curve $(x(t, x_0, t_0), t)$, $t_0 \leq t \leq t_1$. The set D may be either closed or not; it may even consist of a single integral curve.

Let us call the control $u(x, t)$ optimal on D if, for every point (x_0, t_0) in D and any control $\tilde{u}(x, t)$, the following inequality holds

$$\Phi(x(t_1, x_0, t_0)) \leq \Phi(\tilde{x}(t_1, x_0, t_0))$$

where $(\tilde{x}(t_1, x_0, t_0)$ is the integral curve corresponding to $\tilde{u}(x, t)$ that starts from the point x_0, t_0)—not necessarily remaining with D.

The purpose of this paper is to derive and discuss the necessary conditions of optimality for positional controls.

2. NORMAL CONTROL

Retaining the previous notation, let us assume that $u(x, t)$ has a single smooth surface of discontinuity P of dimension n, defined in the neighbourhood of D by the equation $p(x, t) = 0$. The scalar function p is assumed to be continuously differential in the neighbourhood of D with non-zero gradient $\nabla p = (p_x, p_t)$ at points of P. The restrictions of the functions $u(x, t)$ and $f(x, u(x, t), t)$ to the domains $p > 0$ and $p < 0$ will be denoted by $u^{\pm}(x, t)$, $f^{\pm}(x, t)$, respectively. In addition, we shall assume that

$$\dot{p}^{\pm}(x,t) = p_x(x,t)' f^{\pm}(x,t) + p_t(x,t)$$

Here and below we use the prime to denote transposition. The notation $a'b$ will denote the scalar product of (column-)vectors a, b.

We shall call a control $u(x, t)$ normal on the set D if the following conditions are satisfied

1. the functions u^- and u^+ possess extensions, differentiable with respect to x and continuous with respect to t, from the domains $p < 0$ and $p > 0$, respectively, to a small neighbourhood of the surface P, preserving their values on U;
2. in a small neighbourhood of P we have $\dot{p}^- > 0, \dot{p}^+ > 0$ or $\dot{p}^- > 0, \dot{p}^+ \equiv 0$;
3. at each point $(x, t) \in P$ the union of the closures of the sets

$$\{v \in U: p_x(x,t)' f(x,v,t) + p_t(x,t) > 0\}$$
$$\{v \in U: p_x(x,t)' f(x,v,t) + p_t(x,t) < 0\}$$

is the whole of U;

4. a continuous solution $\psi(x, t)$ of the adjoint boundary-value problem

$$\psi_t + \psi_x f(x, u(x,t), t) = -[f_x(x, u(x,t), t) + f_u(x, u(x,t), t) u_x(x,t)]' \psi,$$

$$\psi(x, t_1) = -\Phi_x(x) \tag{2.1}$$

exists on the whole set D and is continuously differentiable in the domains of smoothness of the control.

Finally, a normal control $u(x, t)$ is said to be extremal in D if, at each point $(x, t) \in D$, the following conditions is satisfied

$$\psi(x,t)' f(x,u(x,t),t) = \max_{v \in U} \psi(x,t)' f(x,v,t) \tag{2.2}$$

3. FORMULATION AND DISCUSSION OF THE MAIN RESULT

Theorem. (The positional maximum principle.) An optimal normal positional control on D is extremal on that set.

The proof is postponed to the Appendix. Let us analyse the theorem.

In a domain in which a normal extremal control is smooth, we have

$$[f_u(x, u(x, t), t)u_x(x, t)]'\psi(x, t) = 0$$

Indeed, for any two nearby points (x, t), $(x + \Delta x, t)$ in the smoothness domain of an extremal control we may write, using condition (2.2)

$$\psi(x, t)'[f(x, u(x + \Delta x, t), t) - f(x, u(x, t), t)] \leqslant 0$$

Hence, using the fact that Δx is small and arbitrary, standard arguments yield the desired result.

Thus, in the smoothness domain of an extremal control, the differential equations of boundary-value problem (2.1) may be simplified as follows:

$$\psi_t + \psi_x f(x, u(x, t), t) = -f_x(x, u(x, t), t)'\psi \tag{3.1}$$

We will now show the relation of this theorem to Pontryagin's maximum principle [2] for the problem

$$\begin{gathered} J = \Phi(x(t_1)) \to \min, \quad \dot{x} = f(x, u, t) \\ x(t_0) = x_0, \quad u(t) \in U, \quad t \in [t_0, t_1] \end{gathered} \tag{3.2}$$

for fixed x_0 and t_0 in the class of piecewise-continuous open-loop controls. Assume that an optimal process $u(t)$, $x(t)$ exists in this problem and that the control $u(t)$ has at most one point of discontinuity in the interval (t_0, t_1) (for the extension to any finite number of points of discontinuity see Section 5). Suppose, further, that $\psi(t)$ is a suitable continuous solution of the adjoint system

$$\dot{\psi} = -f_x(x(t), u(t), t)'\psi, \quad \psi(t_1) = -\Phi_x(x(t_1))$$

We will take D to be the integral curve $(x(t), t)$, $t_0 \leqslant t \leqslant t_1$. Set $u(x, t) \equiv u(t)$, $\psi(x, t) \equiv \psi(t)$ in a small neighbourhood of D. Then the control $u(x, t)$ is normal on D by the above theorem, the maximum condition (2.2) holds at each point $(x, t) \in D$, i.e.

$$\psi(t)' f(x(t), u(t), t) = \max_{v \in U} \psi(t)' f(x(t), v, t), \quad t \in [t_0, t_1]$$

Thus, for an optimal open-loop process the theorem yields Pontryagin's maximum principle as formulated in [9].

4. THE RELATION OF THE THEOREM TO DYNAMIC PROGRAMMING

Let $B(x_0, t_0)$ denote the infimum of values of the object functional of problem (3.2). We know that at points where $B(x, t)$ is continuously differentiable it satisfies the fundamental equation of dynamic programming (Bellman's equation)

$$V_t + \min_{v \in U} V_x' f(x, v, t) = 0, \quad V|_{t=t_1} = \Phi(x) \tag{4.1}$$

Let $C \subset R^n \times T$ be a domain in which the boundary-value problem (4.1) has a solution $V(x, t)$ with continuous partial derivatives V_{xx} and V_{tx} and such that a function $u: C \to U$ exists, a differentiable with respect to x, for which the following equality holds identically in $(x, t) \in C$

$$V_t(x,t) + \min_{v \in U} V_x(x,t)' f(x,v,t) = V_t(x,t) + V_x(x,t)' f(x,u(x,t),t) = 0 \tag{4.2}$$

Differentiation of identity (4.2) with respect to x shows that the function

$$\psi(x, t) = -V_x(x, t) \tag{4.3}$$

satisfies the conditions of the adjoint boundary-value problem (2.1). In that situation the extremality of the control $u(x, t)$ follows from (4.2).

The relationship between the adjoint function of the positional maximum principle and the solution of Bellman's equation is not always valid. For example, considering the problem

$$J = x^2(1) \to \min, \ \dot{x} = u, \ |u| \leqslant 1, \ 0 \leqslant t \leqslant 1$$

we find that Bellman's equation

$$V_t - |V_x| = 0, \ V|_{t=1} = x^2$$

has a set of solutions that are continuous on $R \times [0, 1]$ and continuously differentiable (for $x \neq 0$)

$$V(x,t) = \begin{cases} (|x| + t - 1)^2, & |x| > 1 - t \\ F(|x| + t - 1), & |x| \leqslant 1 - t \end{cases}$$

where $F(z)$ is an arbitrary smooth non-decreasing function, $F(0) = F_z(0) = 0$.

The fundamental relations of the positional maximum principle

$$\psi_t + \psi_x u(x, t) = 0, \quad \psi|_{t=1} = -2x, \quad \psi(x, t)u(x, t) = |\psi(x, t)|$$

for the unknown solution $u(x, t)$, together with the continuity condition for $\psi(x, t)$, yield a unique adjoint function

$$\psi(x,t) = \begin{cases} -2(|x| + t - 1)\operatorname{sign} x, & |x| > 1 - t \\ 0, & |x| \leqslant 1 - t \end{cases}$$

If $F(z) \not\equiv 0$, Eq. (4.3) fails to hold in the domain $|x| < 1 - t$.

As this example shows, Bellman's equation may yield extraneous solutions. If one does not appeal to additional considerations and remains within the bounds of dynamic programming, it is impossible to reject these extraneous solutions without calculating the values of $B(x, t)$, i.e. the solutions of the initial optimal control problem with initial data as parameters. Clearly, once the problem becomes more complicated, this becomes practically impossible.

Changing the sign of the object functional in the example, we obtain the problem

$$J = -x^2(1) \to \min, \ \dot{x} = u, \ |u| \leqslant 1, \ 0 \leqslant t \leqslant 1$$

in which the minimum of the object functional

$$B(x, t) = -(|x| - t + 1)^2$$

is not a continuously differentiable function of x at $x = 0$. In that case there is no formal justification for using the classical Bellman equation.

Applying the positional maximum principle to the problem, we obtain a set of relations

$$\begin{gathered} \psi_t + \psi_x u(x, t) = 0, \quad \psi|_{t=1} = -2x \\ \psi(x, t)u(x, t) = |\psi(x, t)| \end{gathered} \tag{4.4}$$

for the unknown control $u(x, t)$. We shall seek $u(x, t)$ as a step function. By (4.4)

$$u_1(x, t) = 1, \quad \psi_1(x, t) = 2(x - t + 1), \quad \text{if } x \geqslant t - 1 \tag{4.5}$$

$$u_2(x, t) = -1, \quad \psi_2(x, t) = 2(x + t - 1), \quad \text{if } x \leqslant 1 - t \tag{4.6}$$

The characteristic curves of the adjoint boundary-value problem corresponding to controls (4.5) and (4.6) cover the domain $|x| \leqslant 1 - t$ twice. Let us single out from each pair of characteristics issuing from an arbitrary fixed point of the domain the characteristic with the least value of the objective functional. As a result we obtain two normal extremal controls

$$u_1(x, t) = 1 \text{ on the set } x \geqslant 0 \tag{4.7}$$

$$u_2(x, t) = -1 \text{ on the set } x \leqslant 0 \tag{4.8}$$

It is clear that each of these is optimal on the appropriate set.

Thus, in the above problem the positional maximum principle defines two domains in the space $R \times [0, 1]$ and extremal controls in those domains.

5. EXTENSION OF THE THEOREM

The proof of the theorem presented in the appendix extends to the case in which the control has several surfaces of discontinuity, provided one modifies the notion of normality as follows: Suppose one has a domain $D \subset R^n \times T$ and a control $u(x, t)$ with $m \geqslant 1$ surfaces of discontinuity $P_1, P_2, \ldots, P_m$ on D, of dimension n, defined by the equations

$$p_1(x, t) = 0, \quad p_2(x, t) = 0, \ldots, \quad p_m(x, t) = 0$$

respectively. Each function p_i: $D \to R$, $i = 1, 2, \ldots, m$, is assumed to be continuously differentiable in the neighbourhood of F, with non-zero gradient at the points of P_i. Let Q_i denote a small neighbourhood of the surface P_i and Q_i^-, Q_i^+ the intersections of Q_i with the domains $p_i < 0, p_i > 0$, respectively.

We will call a control $u(x, t)$ normal in D if the following conditions hold:

1. the surfaces $P_1, P_2, \ldots, P_m$ do not intersect one another on D;
2. the restrictions of $u(x, t)$ to the half-neighbourhoods Q_i^-, Q_i^+

$$u|_{Q_i^-} = u_i^-, \quad u|_{Q_i^+} = u_i^+$$

can be continued as functions differentiable with respect to x and continuous in t to Q_i, conserving their values in U for each fixed $i = 1, 2, \ldots, m$;

3. in Q_i the derivatives

$$\dot{p}_i^{\pm}(x,t) = p_{ix}(x,t)' f(x, u_i^{\pm}(x,t), t) + p_{it}(x,t)$$

satisfy the conditions

$$\dot{p}_i^- > 0, \quad \dot{p}_i^+ > 0, \quad i = 1, 2, \ldots, m-1$$
$$\dot{p}_m^- > 0, \quad \dot{p}_m^+ \equiv 0$$

4. at each point $(x, t) \in P_m$ the union of the closures of the sets

$$\{v \in U\colon\ p_{mx}(x, t)' f(x, v, t) + p_{mt}(x, t) > 0\}$$
$$\{v \in U\colon\ p_{mx}(x, t)' f(x, v, t) + p_{mt}(x, t) < 0\}$$

is the whole of U;

5. the adjoint boundary-value problem (2.1) has a continuous solution, defined on the whole of D.

With normal controls thus defined, the positional maximum principle remains valid in the same formulation as before.

6. EXAMPLES

The use of the positional maximum principle will be illustrated by two familiar examples.

Example 1 (thc linear problem [10])

$$J = c'x(t_1) \to \min, \quad \dot{x} = A(t)x + b(u, t)$$
$$u(x, t) \in U, \quad (x, t) \in R^n \times T$$

where the function $\Phi(x) = c'x$ is linear and the function $f(x, u, t) = A(t)x + b(u, t)$ satisfies the assumptions listed previously.

Define functions ψ: $T \to R^n$ and u: $T \to U$ by

$$\dot{\psi} = -A(t)'\psi, \quad \psi(t_1) = -c, \quad u(t) = \arg\max_{v \in U} \psi(t)' b(v, t)$$

and assume that $u(t)$ is piecewise-continuous in the interval T. Then the pair

$$u(x, t) \equiv u(t), \quad \psi(x, t) \equiv \psi(t)$$

defined on the set $R^n \times T$, satisfies the normality conditions of Sections 4 and the assumptions of the positional maximum principle. A direct calculation of the objective functional and the evaluation of its minimum show that $u(x, t)$ is indeed an optimal control on the set $R^n \times T$.

Example 2 (the linear quadratic problem: analytical regulator design [11])

$$J = \tfrac{1}{2} y(t_1) \to \min, \quad \dot{x} = A(t)x + B(t)u$$
$$\dot{y} = x' P(t)x + u' Q(t)u, \quad u(x, y, t) \in U, \quad (x, y, t) \in R^{n+1} \times T$$

where $A(t)$, $B(t)$, $P(t)$, $Q(t)$ are continuous matrices of appropriate dimension on T, $P(t)$ is symmetric and positive semidefinite, $Q(t)$ is symmetric and positive definite for all $t \in T$ and U is an open sphere of sufficiently large radius in R^r.

Since the domain of the control is open, the fundamental relations of the positional maximum principle become

$$\psi_t + \psi_x(Ax + Bu) + (x'Px + u'Qu)\psi_y + A'\psi + 2\chi Px = 0$$
$$\chi_t + \chi_x'(Ax + Bu) + (x'Px + u'Qu)\chi_y = 0$$
$$\psi|_{t=t_1} = 0, \quad \chi|_{t=t_1} = -\tfrac{1}{2}, \quad B'\psi + 2\chi Qu = 0$$

(for brevity we have omitted the arguments, $\chi = \psi_{n+1}$). These conditions are satisfied in the domain $D = R^{n+1} \times T_1$ if we put

$$\psi(x, y, t) = K(t)x, \quad \chi(x, y, t) = -\tfrac{1}{2}$$
$$u(x, y, t) = Q^{-1}(t)B(t)'K(t)x$$

and define $K(t)$ to be a solution on $T_1 \subset T$ of the matrix Riccati equation

$$\dot{K} + KA(t) + A(t)'K + KB(t)Q^{-1}(t)B(t)'K - P(t) = 0, \quad K(t_1) = 0$$

This solution of the problem of analytical regulator design is identical with those produced by the variational method [11] and by dynamic programming [3].

7. SYNTHESIS OF PIECEWISE-CONSTANT CONTROLS

Suppose that the function $u \to f(x, u, t)$ in problem (1.1) is affine and that U is a finite set of points or a polyhedron. Then an extremal control $u(x, t)$ is generally a step function. To find it, one can use an analogue of the method of retrograde motion [2]. We will describe the main operations to be performed at step k, $k = 1, 2, \ldots$.

Put $B = R^n \times (0, \infty)$ and suppose that the following are known for $k \geq 0$: a vector u^k in U, a set $D_k \subset B$ with boundary ∂D_k and a function ψ^k: $\partial D_k \to R^n$.

Let us find a continuous solution $\psi(x, w, t)$ of the boundary-value problem (see (3.1))

$$\psi_t + \psi_x f(x,w,t) = -f_x(x,w,t)'\psi, \quad \psi|_{\partial D_k} = \psi^k(x,t) \tag{7.1}$$

with vector parameter $w \in R^r$. Furthermore, define $w = u^{k+1} \in U$ and a maximal set $D_{k-1} \subset B \backslash \cup_{i=0}^{k} D_i$ by the conditions

$$\psi(x,u^{k+1},t)' f(x,u^{k+1},t) = \max_{v \in U} \psi(x,u^{k+1},t)' f(x,v,t), \; (x,t) \in D_{k+1} \tag{7.2}$$

On the set D_{k+1} we put

$$u(x, t) = u^{k+1}, \quad \psi^{k+1}(x, t) = \psi(x, u^{k+1}, t)$$

The process is continued if $D_{k+1} = \varnothing$. For $k = 0$ we define $D_0 = R^n \times (t_1, \infty)$, $\psi^0(x, t) = -\Phi_x(x)$, letting u^0 be any point of U.

The procedure needs some explanation. If the integral curves of the equation $\dot{x} = f(x, w, t)$ cut the boundary ∂D_k without tangency, the boundary-value problem (7.1) is locally solvable by the method of characteristics [8], which extends in a natural way to the case of a system of first-order linear partial differential equations. To carry out the operation (7.2) and construct a maximal domain D_{k+1} it is necessary to solve non-linear equations and inequalities—this involves certain difficulties, reflecting the complexity of the control synthesis problem itself.

The synthesis procedure may be illustrated by a simple example

$$J = -x^4(1) \to \min, \quad \dot{x} = u, \quad |u| \leq 1, \quad 0 \leq t \leq 1$$

Here, as in Example 2 above, the minimum of the objective functional

$$B(x, t) = -(|x| - t + 1)^4$$

is a continuous function, continuously differentiable with respect to x, on $R \times [0, 1]$ everywhere except on the straight-line segment $x = 0$, $0 \leq t \leq 1$. Formally, therefore, the conditions for application of the classical Bellman equation are not satisfied. At the first stage of the synthesis procedure, a literal repetition of the arguments in Section 4 yields two normal extremal controls (4.7), (4.8) with corresponding adjoint functions

$$\psi_1(x, t) = 4(x - t + 1)^3, \quad x \geq 0$$
$$\psi_2(x, t) = 4(x + t - 1)^3, \quad x \leq 0$$

Note that if one replaces the unimodal objective function $\Phi(x) = -x^4$ in this example by the multimodal function

$$\Phi(x) = \left(\cos\frac{\pi x}{2} + \left|\cos\frac{\pi x}{2}\right|\right)^4$$

then the minimum function $B(x, t)$ of the objective functional will not be continuously differentiable on the denumerable set of straight segments $x = 4k$, $0 \leq t \leq 1$, $k = 0, \pm 1, \pm 2, \ldots$. Hence the problems involved in justifying the use of Bellman's equation remain. As the objective function is periodic, it is sufficient to consider the synthesis procedure on the set $[x - 4k] \leq 2$ for fixed $k = 0, \pm 1, \pm 2, \ldots$, where it produces a true result.

APPENDIX: PROOF OF THE THEOREM

Suppose the control $u(x, t)$ has a unique surface of discontinuity $P \subset D$ on D and is normal in the sense of the definition of Section 2. To fix our ideas, suppose that in a small neighbourhood Q of P the conditions $p^- > 0$, $p^+ \equiv 0$ are satisfied. (The simpler case $p^- > 0$, $p^+ > 0$ receives analogous treatment.) Fix an arbitrary point (x_0, t_0) of D in the domain $p < 0$ and consider the integral curve leaving it under the action of $u(x, t)$, say

$(x(t) \equiv x(t, x_0, t_0), t)$. Assume that the curve $(x(t), t)$ reaches the surface P at time $\theta \in (t_0, t_1)$ and remains on P for $\theta \leq t \leq t_1$. The other possibilities (the curve does not cut P or lies wholly on P) are treated in exactly the same way.

Variation of the control and the trajectory. Let us determine the varied control $\tilde{u}(x, t)$ and trajectory $\tilde{x}(t) \equiv \tilde{x}(t, x_0, t_0)$ of system (1.1).

Choose arbitrary fixed $\upsilon \in U$, $\tau \in [t_0, t_1)$, $\tau \neq \theta$ and a small $\varepsilon > 0$. Define $\tilde{u}(x, t) = \upsilon$ if $t \in [\tau, \tau + \varepsilon)$ and

$$\tilde{u}(x,t) = \begin{cases} u^-(x,t), & \tilde{p}(x,t) < 0 \\ u^+(x,t), & \tilde{p}(x,t) > 0 \end{cases} \tag{A.1}$$

if $t \notin [\tau, \tau + \varepsilon)$, where, by definition

$$\tilde{p}(x,t) = p(x,t) + \varepsilon\delta p(t) \tag{A.2}$$

Assume that the function δp: $T \to R$ is smooth and of fixed sign on T. By conditions (A.1) and (A.2) and the choice of δp, the varied surface $\tilde{P}$ (the surface of discontinuity of the control $\tilde{u}(x, t)$ with the equation $\tilde{p}(x, t) = 0$) is shifted uniformly from 0 into one of the domains $p > 0, p < 0$.

We shall now describe the varied trajectory $\tilde{x}(t)$. By analogy with [7, 12], one can prove the formula

$$\tilde{x}(t) = x(t) + \varepsilon\delta x(t) + o(\varepsilon), \quad t_0 \leq t \leq t_1 \tag{A.3}$$

where $\delta x(t)$, for $t \neq \tau$, $t \neq \theta$, is a continuous solution of the variations equation

$$(\delta x)^{\cdot} = [f_x(x(t),u(x(t),t),t) + f_u(x(t),u(x(t),t),t)u_x(x(t),t)]\delta x, \quad \delta x(t_0) = 0 \tag{A.4}$$

satisfying the above jump conditions at times τ, θ, and $o(\varepsilon)$ is a vector remainder term of higher order of smallness than ε uniformly in t in the interval $[t_0, t_1]$. On the right of Eqs (A.4) we have [7]

$$\begin{aligned} &u(x(t),t) = u^-(x(t),t), \quad u_x(x(t),t) = u_x^-(x(t),t), \quad t_0 \leq t < \theta \\ &u(x(t),t) = u^+(x(t),t), \quad u_x(x(t),t) = u_x^+(x(t),t), \quad \theta \leq t \leq t_1 \end{aligned} \tag{A.5}$$

For brevity, let us put

$$\begin{aligned} &\delta x(s\pm) = \delta x(s \pm 0) \\ &\Delta_\upsilon f|_\tau = f(x(\tau), \upsilon, \tau) - f(x(\tau), u(x(\tau), \tau), \tau) \\ &\Delta\dot{x}(\theta) = f^-(x(\theta), \theta) - f^+(x(\theta), \theta) \\ &\delta\theta = -[\delta p(\theta) + p_x(x(\theta), \theta)'\delta x(\theta-)]/\dot{p}^-(x(\theta), \theta) \end{aligned}$$

If the inequalities

$$\tau > \theta, \quad p_x(x(\tau), \tau)'f(x(\tau), \upsilon, \tau) + p_t(x(\tau), \tau) > 0 \tag{A.6}$$

hold, or if $\tau < \theta$, the jump conditions become

$$\begin{aligned} &\delta x(\tau+) = \delta x(\tau-) + \Delta_\upsilon f|_\tau \\ &\delta x(\theta+) = \delta x(\theta-) + \delta\theta\Delta\dot{x}(\theta) \end{aligned} \tag{A.7}$$

But if

$$\tau > \theta, \quad p_x(x(\tau), \tau)'f(x(\tau), \upsilon, \tau) + p_t(x(\tau), \tau) < 0 \tag{A.8}$$

then, instead of (A.7), we have

$$\begin{aligned} &\delta x(\tau+) = \delta x(\tau-) + \Delta_\upsilon f|_\tau + \delta\tau\Delta\dot{x}(\tau) \\ &\delta\tau = -[\delta p(\tau) + p_x(x(\tau),\tau)'\Delta_\upsilon f|_\tau]\dot{p}^-(x(\tau),\tau) \\ &\delta x(\theta+) = \delta x(\theta-) + \delta\theta\Delta\dot{x}(\theta) \end{aligned} \tag{A.9}$$

A few explanations are in order. Up to terms of order higher than ε, the variations equation (A.4) and jump conditions (A.7) and (A.9) describe the principal part $\varepsilon\delta x(t)$ of the increment $\tilde{x}(t)-x(t)$ of the trajectories at the time the control is subject to a spiked variation and the trajectories cut the surfaces of discontinuity P and $\tilde{P}$. The variation of the surface P introduces the arbitrarily chosen function $\delta p(t)$ into the jump conditions. As will be evident, this entails the need for another necessary optimum condition of the equality type—the continuity of the solution of the adjoint system on the surface of discontinuity of the control.

For points $\upsilon \in U$ satisfying the second inequality of (A.6), at the time τ the control is subject to a spiked variation of the trajectory $\tilde{x}(t)$ hits the surface $\tilde{P}$ in the domain $\tilde{p} > 0$, and it remains there for $t > \tau$. In the case of (A.8) one has an analogous encounter in the domain $\tilde{p} < 0$ and the varied trajectory subsequently returns to $\tilde{P}$ beginning at time $\tau + \varepsilon\delta\tau + o(\varepsilon)$. For other points of U it is generally impossible to describe the function $\tilde{x}(t)$ uniquely to within the accuracy needed by formula (A.3). Given condition 3 in the definition of normality, this does not affect the generality of the necessary optimum conditions. Finally, one should note that if the definitions of the right-hand sides of the differential equations are extended to P, then

$$f(x, u(x, t), t)|_P = f^+(x, t)$$

and so one must assume in formulae (A.7) and (A.9) that for $\tau > \theta$

$$\Delta_\upsilon f|_\tau = f(x(\tau), \upsilon, \tau) - f^+(x(\tau), \tau)$$

The maximum principle. It follows from our assumption about the optimality of the control $u(x, t)$ and from representation (A.3) of the varied trajectory that

$$\Phi(x(t_1)) \leqslant \Phi(x(t_1) + \varepsilon\delta x(t_1) + o(\varepsilon))$$

Hence, since ε is small and positive, we obtain

$$\Phi_x(x(t_1))'\delta x(t_1) \geqslant 0 \tag{A.10}$$

In order to express this inequality in terms of the parameters of the variation of the control, we use the solution $\psi(x, t)$ of the adjoint boundary-value problem. Let $\psi^-(x, t)$, $\psi^+(x, t)$ denote the restrictions of this solution to the domains $p < 0$, $p > 0$, respectively. In view of the normality of the control $u(x, t)$ and the special features of the method of characteristics [8], the functions $\psi^\pm(x, t)$ may be extended smoothly to a small neighbourhood of the surface P so that condition (2.1) remains valid when $u(x, t)$ is replaced by the extensions of $u^\pm(x, t)$. Consequently, the compound function

$$\Psi(t) = \begin{cases} \psi^-(x(t), t), & t_0 \leqslant t \leqslant \theta \\ \psi^+(x(t), t), & \theta \leqslant t \leqslant t_1 \end{cases}$$

is defined and continuous in the segment $[t_0, t_1]$, and it satisfies the following system of ordinary differential equations in the intervals (t_0, θ) and (θ, t_1)

$$\begin{aligned} \dot{\Psi} = &-[f_x(x(t), u(x(t), t), t) + \\ &+ f_u(x(t), u(x(t), t), t)u_x(x(t), t)]'\Psi, \quad \Psi(t_1) = -\Phi_x(x(t_1)) \end{aligned} \tag{A.11}$$

where $u(x(t), t)$, $u_x(x(t), t)$ are understood in the sense of (A.5).

In the intervals in which the solutions of Eqs (A.4) and (A.11) are smooth, the function $\psi(t)'\,\delta x(t)$ is constant [2], and therefore its increment at the ends of the interval (t_0, t_1) equals the sum of the jumps at the points τ, θ

$$\Psi'\delta x\Big|_{t_0}^{t_1} = \Psi'\delta x\Big|_{\tau-}^{\tau+} + \Psi'\delta x\Big|_{\theta-}^{\theta+}$$

Taking the initial data (A.4), (A.11) and jump conditions (A.7) into account, we deduce from this and from (A.10) that

$$\Phi_x(x(t_1))'\delta x(t_1) = -\Psi(\tau+)'\Delta_\upsilon f|_\tau - \mu\delta p(\theta) + [\Psi(\theta-) - \Psi(\theta+) - \mu p_x(x(\theta), \theta)]'\delta x(\theta-) \geqslant 0$$

where

$$\mu = -\Psi(\theta+)'\Delta\dot{x}(\theta)/\dot{p}^-(x(\theta), \theta)$$

By the continuity of $\psi(t)$ and the arbitrariness of $\delta p(\theta)$, we conclude from the inequality that

$$\Psi(\theta)'\Delta\dot{x}(\theta) = 0 \tag{A.12}$$

$$\Psi(\tau)'\Delta_v f|_\tau \leqslant 0 \tag{A.13}$$

In case (A.8), we also conclude, besides (A.12) and (A.13), that

$$\Psi(\tau)'\Delta\dot{x}(\tau) = 0 \tag{A.14}$$

Note that these last three conditions are not independent: Eqs (A.12) and (A.14) follow from (A.13) by letting $\tau \to \theta$. Since inequality (A.13) holds for any trajectory $x(t)$ with the properties described above, and by continuity it remains true for all $\tau \in [t_0\, t_1]$, $\upsilon \in U$, it follows that at points $(x, t) \in D, p(x, t) < 0$

$$\psi(x, t)'[f(x, \upsilon, t) - f(x, u(x, t), t)] \leqslant 0$$

Similar arguments yield the same inequality for the other points of D. Consequently, the optimal control is extremal in D, completing the proof of the theorem.

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